Last time: - Spectral Thm for Hilbert spaces

- Fourier transforms

What does the Fourier transform look like for $\mathbb{R}$?

$G = \mathbb{R}$ Haar measure = Lebesgue measure

Recall that the Fourier transform is a unitary map

$$U : L^2(G) \to L^2(\hat{G})$$

where

$$\hat{G} = \{\phi : G \to U(1) \mid \underset{\text{grp hom}}{\phi \text{ is continuous}}\}$$

What is $\hat{\mathbb{R}}$?

$$\hat{\mathbb{Z}} = \{ \chi_\lambda \mid \lambda \in U(1) \}$$

where $\chi_\lambda(x) = \lambda^x$.

$$\hat{\mathbb{R}} = \{ \chi_p \mid p \in \mathbb{R} \}$$

where $\chi_p(x) = \exp\left(\frac{ipx}{\hbar} \right)$

where $\hbar = 1.05 \times 10^{-34} \, kg \frac{m^2}{s}$

So we usually identify $\hat{\mathbb{R}}$ with $\mathbb{R}$

via $p \mapsto \chi_p$

So if $\psi \in L^2(\mathbb{R})$, then the

Fourier transform $U\psi = \tilde{\psi} \in L^2(\hat{\mathbb{R}})$.

If it happens that $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$,

then the Fourier transform is given by

$$\tilde{\psi}(\chi) = \int_{\mathbb{R}} \overline{\chi(x)} \, \psi(x) \, dx$$

If we identify $\hat{\mathbb{R}}$ with $\mathbb{R}$, this gives

$$\tilde{\psi}(p) = \int_{\mathbb{R}} \exp\left(\frac{-ipx}{\hbar}\right) \psi(x)\, dx .$$

The position operator on $L^2(\mathbb{R})$ is denoted $\hat{X}$ and takes

$$\psi \in L^2(\mathbb{R}) \quad \text{to} \quad \hat{X}\psi(x) = x\psi(x)$$

so $\hat{X}$ is diagonal, and $\hat{X} = L_x$.

Issue: if $\psi = \frac{1}{\sqrt{1+x^2}}$

then $\hat{X}\psi = \frac{x}{\sqrt{1+x^2}}$

But $\int |\hat{X}\psi|^2\, dx = \int \frac{x^2}{1+x^2}\, dx = \infty$.

So $\hat{X}$ might send L^2 functions out of L^2.

We fix this by redefining the word operator.

Now, an operator on V is a map from a subspace of V, called its domain, to V.

We denote the domain of an operator L to be $\mathrm{Dom}(L)$.

Such operators are also called unbounded operators.

We can take

$\mathrm{Dom}(\hat{X})$ to be

$$\{ \psi \in L^2(\mathbb{R}) \mid x\psi \in L^2(\mathbb{R}) \}$$

It turns out that $\overline{\mathrm{Dom}(\hat{X})} = L^2(\mathbb{R})$

We usually require that the domain of an unbounded operator is a dense subspace of our Hilbert space.

Similarly, the momentum operator is given on $L^2(\hat{\mathbb{R}})$ by

$$\hat{P}\,\tilde{\psi} = p\,\tilde{\psi}$$
$$= p \mapsto p\,\tilde{\psi}(p)$$

To transfer $\hat{P}$ to the position picture, we view it as $\mathcal{U}^{-1}\hat{P}\,\mathcal{U}$, where $\mathcal{U}$ is the Fourier transform $L^2(\mathbb{R}) \to L^2(\hat{\mathbb{R}})$.

We can determine explicitly what $\hat{P}$ looks like in the position picture, using the fact that our translation operators are diagonal in $L^2(\hat{\mathbb{R}})$.

Given $\psi \in L^2(\mathbb{R})$, we have

$$\mathcal{U}^{-1}\hat{P}\,\mathcal{U}\,\psi = \mathcal{U}^{-1}\hat{P}\,\tilde{\psi}$$
$$= \mathcal{U}^{-1}\,p\,\tilde{\psi}$$

$$= U^{-1} \frac{\hbar}{i} \frac{d}{dt} \exp\left(\frac{i\hat{P}t}{\hbar}\right) \tilde{\psi} \Big|_{t=0}$$

$$= U^{-1} \frac{\hbar}{i} \frac{d}{dt} U T_t U^{-1} \tilde{\psi} \Big|_{t=0}$$

$$= \frac{\hbar}{i} \frac{d}{dt} T_t \psi \Big|_{t=0}$$

$$= \frac{\hbar}{i} \frac{d}{dt} \psi(x+t) \Big|_{t=0}$$

$$= \frac{\hbar}{i} \frac{d}{dx} \psi$$

the function sending $x \in \mathbb{R}$ to $\frac{\hbar}{i} \frac{d\psi(x)}{dx}$

So in the position picture

$$\hat{P} = -i\hbar \frac{d}{dx}$$

with $\mathrm{Dom}(\hat{P}) = \{ \psi \in L^2(\mathbb{R}) \mid \psi \text{ is differentiable and } \frac{d}{dx}\psi \in L^2(\mathbb{R}) \}$.

Just like in finite dimensions, the Hamiltonian is given by

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X}) = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

mass, just some positive real number

where $V: \mathbb{R} \to \mathbb{R}$ is some potential function we still want to solve

$$i\frac{d}{dt}U(t) = \hat{H}\,U(t).$$

to determine what time evolution does to the "particle on $\mathbb{R}$ with mass m and potential function V" quantum system.

We can still solve this using

$$U(t) = \exp(-it\hat{H})$$

But this expression might not make sense for an unbounded operator. However if $\hat{H}$ is normal (which it is; in fact $\hat{H}$ is self-adjoint), then we can make sense of the exponential using the spectral theorem for unbounded operators.

This theorem tells us we can diagonalize even unbounded normal operators.

An unbounded diagonal operator on $L^2(X)$ is one of the form L_f for some measurable function $f: X \to \mathbb{C}$ (so we no longer restrict ourselves to $f \in L^\infty(X)$ like we did in the bounded case.)

We can easily exponentiate the unbounded diagonal operator L_f; we just define

$$\exp(-it L_f) := L_{\exp(-itf)} .$$